

# Characterization of Minimum Cycle Basis in Weighted Partial 2-trees \*

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## Abstract

For a weighted outerplanar graph, the set of lex short cycles is known to be a minimum cycle basis [Inf. Process. Lett. 110 (2010) 970-974]. In this work, we show that the set of lex short cycles is a minimum cycle basis in weighted partial 2-trees (graphs of treewidth two) which is a superclass of outerplanar graphs.

## 1 Introduction

A cycle basis is a compact description of the set of all cycles of a graph and has various applications including the analysis of electrical networks [6]. Let  $G = (V(G), E(G))$  be an edge weighted graph and let  $m = |E(G)|$  and  $n = |V(G)|$ . A *cycle* is a connected graph in which the degree of every vertex is two. An *incidence vector*  $x$ , indexed by  $E(G)$  is associated with every cycle  $C$  in  $G$ , where for every edge  $e \in E(G)$ ,  $x_e$  is 1 if  $e \in E(C)$  and 0 otherwise. The *cycle space* of  $G$  is the vector space over  $\mathbb{F}_2^m$  spanned by the incidence vectors of cycles in  $G$ . A *cycle basis* of  $G$  is a minimum set of cycles whose incidence vectors span the cycle space of  $G$ . The weight of a cycle  $C$  is the sum of the weights of the edges in  $C$ . A cycle basis  $B$  of  $G$  is a *minimum cycle basis* (MCB) if the sum of the weights of the cycles in  $B$  is minimum. A minimum cycle basis of  $G$  is denoted by  $MCB(G)$ .

**Motivation:** For a weighted graph  $G$ , Horton has identified a set  $\mathcal{H}$  of  $O(mn)$  cycles and has shown that a minimum cycle basis of  $G$  is a subset of  $\mathcal{H}$  [5]. Liu and Lu have shown that the set of *lex short cycles* (defined later) is a minimum cycle basis in weighted outerplanar graphs [8]. We generalize this result for partial 2-trees which is a superclass of outerplanar graphs.

**Our contribution:** The following are the main results in this work.

**Theorem 1.1** *Let  $G$  be a weighted partial 2-tree on  $n$  vertices and  $m$  edges. Then the number of lex short cycles in  $G$  is  $m - n + 1$ .*

**Theorem 1.2** *For a weighted partial 2-tree  $G$ , the set of lex short cycles is a minimum cycle basis.*

**Related work:** The characterization of graphs using cycle basis was initiated by MacLane [9]. In particular, MacLane showed that a graph  $G$  is planar if and only if  $G$  contains a cycle basis  $B$  such that each edge in  $G$  appears in at most two cycles of  $B$ . However, he referred to a cycle basis as a *complete independent set of cycles*. Formally, the concept of cycle space in graphs was introduced in [3] after four decades. Later, it was characterized that a planar 3-connected graph  $G$  is a Halin graph if and only if  $G$  has a planar basis  $B$  such that each cycle in  $B$  has an external edge [12]. There after, it was shown that every 2-connected outerplanar graph has a unique MCB [7]. Subsequently, it was proven that Halin graphs that are not necklaces have a unique MCB [11].

\*A preliminary version of this paper appeared as *Characterization of Minimum Cycle Basis in Weighted Partial 2-trees* in the proceedings of CTW 2012

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The first polynomial time algorithm for finding an MCB was given by Horton [5]. Since then, many improvements have taken place on algorithms related to minimum cycle basis and its variants. A detailed survey of various algorithms, characterizations and the complexity status of cycle basis and its variants was compiled by Kavitha et al.[6]. The current best algorithm for MCB runs in  $O(m^2n/\log n)$  time and is due to Amaldi et al.[1].

**Graph preliminaries:** In this paper, we consider only simple, finite, connected, undirected and weighted graphs. We refer [13] for standard graph theoretic terminologies. Let  $G$  be an edge weighted graph. Let  $X \subseteq V(G)$ .  $G - X$  denotes the graph obtained after deleting the set of vertices in  $X$  from  $G$ .  $G[X]$  denotes the subgraph induced by vertices in  $X$ .  $X$  is a *vertex separator* if  $G - X$  is disconnected. A *component* of  $G$  is a maximal connected subgraph.  $K_3$  denotes a cycle on 3 vertices and  $K_2$  denotes an edge.  $K_{2,3}$  is a complete bipartite graph  $(V_1, V_2)$  such that  $|V_1| = 2, |V_2| = 3$ . A graph is *planar* if it can be drawn on the plane without any edge crossings. A planar graph is *outerplanar* if it can be drawn on the plane such that all of its vertices lie on the boundary of its exterior region. A 2-tree is defined inductively as follows:  $K_3$  is a 2-tree; if  $G'$  is a 2-tree and  $G = G' \cup \{v\}$  is such that  $N_G(v)$  forms a  $K_2$  in  $G$ , then  $G$  is a 2-tree. A graph is a *partial 2-tree* if it is a subgraph of a 2-tree. Alternatively, a graph of treewidth (defined in [10]) two is a *partial 2-tree*. An *H-subdivision* (or subdivision of  $H$ ) is a graph obtained from a graph  $H$  by replacing edges with pairwise internally vertex disjoint paths.

## 2 MCB in Weighted Partial 2-trees

For a weighted partial 2-tree  $G$  associated with a weight function  $w : E(G) \rightarrow \mathbb{N}$ , we show that the set of lex short cycles (defined below) in  $G$  is an  $MCB(G)$ . The notion of lex shortest path and lex short cycle is presented from [4]. For a totally ordered set  $S$ ,  $\min(S)$  denotes the minimum element in  $S$ . For a graph  $G$ , let  $V(G)$  be a totally ordered set. A path  $P(u, v)$  between two distinct vertices  $u$  and  $v$  is *lex shortest path* if for all the paths  $P'$  between  $u$  and  $v$  other than  $P$ , exactly one of the following three conditions hold: 1)  $w(P') > w(P)$  2)  $w(P') = w(P)$  and  $|E(P')| > |E(P)|$  3)  $w(P') = w(P)$ ,  $|E(P')| = |E(P)|$  and  $\min(V(P') \setminus V(P)) > \min(V(P) \setminus V(P'))$ , where  $w(P) = \sum_{e \in E(P)} w(e)$ . The lex shortest path between any two vertices  $u$  and  $v$  is unique and is denoted by  $lsp(u, v)$ . A cycle  $C$  is *lex short* if for every two vertices  $u$  and  $v$  in  $C$ ,  $lsp(u, v) \subset C$ . The set of lex short cycles of  $G$  is denoted by  $LSC(G)$ . For a subgraph  $G_1$  of  $G$ , the total order of  $V(G_1)$  is the order induced by the total order of  $V(G)$ . We use  $lsp_{G_1}(x, y)$  to denote the lex shortest path between vertices  $x$  and  $y$  in  $G_1$ . We use the following lemmas from the literature.

**Lemma 2.1** ([4]) *For a simple weighted graph  $G$ ,  $LSC(G)$  contains an  $MCB(G)$ .*

**Lemma 2.2** ([8]) *For a simple weighted outerplanar graph  $G$ ,  $|LSC(G)| = m - n + 1$ .*

We present the following lemmas and theorems that are required to prove our main result.

**Lemma 2.3** *Let  $G$  be a partial 2-tree and  $\{u, v\}$  be a vertex separator in  $G$ . Let  $P$  be the lex shortest path between  $u$  and  $v$ . There exist one component  $H$  in  $G - \{u, v\}$  such that  $V(P) \cap V(H) = \emptyset$  and  $E(P) \cap E(H) = \emptyset$ .*

**Proof** If  $P = (u, v)$ , then none of the components in  $G - \{u, v\}$  contain  $V(P)$  and  $E(P)$ . If  $P = (u, x, v)$ , then no component in  $G - \{u, v\}$  contain  $E(P)$  and exactly one component in  $G - \{u, v\}$  contains  $x$ . If  $P$  is not captured by these two cases, then  $P$  has at least three edges. If  $|E(P)| \geq 3$ , then exactly one component in  $G - \{u, v\}$  that contains  $P - \{u, v\}$ . Since  $\{u, v\}$  is a vertex separator in  $G$ , the number of components in  $G - \{u, v\}$  is at least two. Therefore, there exist a component  $H$  in  $G - \{u, v\}$  such that  $V(P) \cap V(H) = \emptyset$  and  $E(P) \cap E(H) = \emptyset$ .  $\square$

**Lemma 2.4** *Let  $G$  be a partial 2-tree that is not outerplanar. Then there exists a  $K_{2,3}(\{u, v\}, \{x, y, z\})$ -subdivision in  $G$  such that  $G - \{u, v\}$  contains at least three components.*

**Proof** A graph is outerplanar if and only if it contains no subgraph that is a subdivision of  $K_4$  or  $K_{2,3}$  [2]. Since a partial 2-tree does not contain a subdivision of  $K_4$ , a partial 2-tree is outerplanar if and only if it does not contain a subdivision of  $K_{2,3}$ . Consider a  $K_{2,3}(\{u, v\}, \{x, y, z\})$ -subdivision in  $G$ . Assume that  $G - \{u, v\}$  has at most two components. Then there exist a path in  $G - \{u, v\}$  between two vertices in  $\{x, y, z\}$  which does not go through the other vertex. Without loss of generality, we assume that  $x$  and  $y$  are those two vertices and  $z$  is the other vertex. Such a path between  $x$  and  $y$  is shown as a dotted path in Figure 1. It follows that there are six internal vertex disjoint paths in  $G$ , namely  $P(x, u)$ ,  $P(x, v)$ ,  $P(y, u)$ ,  $P(y, v)$ ,  $P(x, y)$  and  $P(u, v)$  via  $z$ . Thus, there is a  $K_4$ -subdivision on the vertex set  $\{u, v, x, y\}$  in  $G$ . This is a contradiction that  $G$  is a partial 2-tree. Therefore,  $\{u, v\}$  is a vertex separator in  $G$  whose removal gives at least three components.  $\square$

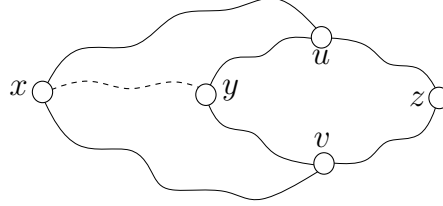


Figure 1: A  $K_4$ -subdivision on the vertex set  $\{u, v, x, y\}$

**Lemma 2.5** Let  $G'$  be a weighted subgraph of a weighted graph  $G$ . Let  $P$  be a path and  $C$  be a cycle contained both in  $G$  and  $G'$ .

- (a). If  $P$  in  $G$  is lex shortest, then  $P$  in  $G'$  is lex shortest.
- (b). If  $C \in LSC(G)$ , then  $C \in LSC(G')$ .

**Proof** Suppose if the path  $P$  in  $G'$  is not lex shortest, then the path  $P$  in  $G$  would not be lex shortest. Hence, the  $P$  in  $G'$  is lex shortest.

Recall that  $C$  is a lex short cycle if for every  $x, y \in V(C)$ ,  $lsp(x, y)$  is contained in  $C$ . Since  $C$  is in  $G'$  and for every  $x, y \in C$ ,  $lsp(x, y)$  is same as  $lsp_{G'}(x, y)$ ,  $C$  is a lex short cycle in  $G'$ .  $\square$

**Lemma 2.6** The intersection of two lex shortest paths is either empty or a lex shortest path.

**Proof** Consider two lex shortest paths  $lsp(x, y)$  and  $lsp(u, v)$  in  $G$ . Let  $G' = (V(G'), E(G'))$ , where  $V(G') = V(lsp(x, y)) \cap V(lsp(u, v))$ ,  $E(G') = E(lsp(x, y)) \cap E(lsp(u, v))$ . Suppose  $V(G') \neq \emptyset$  and  $G'$  is not a path, then we have at least two maximal paths  $P(a, b)$  and  $P(a', b')$  which are common to both  $lsp(x, y)$  and  $lsp(u, v)$ , where  $b \neq a'$ . Consequently, the path  $P_1(b, a')$  contained in  $lsp(x, y)$  and the path  $P_2(b, a')$  contained in  $lsp(u, v)$  are different. Since a subpath of a lex shortest path is a lex shortest path, both  $P_1$  and  $P_2$  are lex shortest paths between  $b$  and  $a'$ ; a contradiction to the fact that for any two vertices in a graph, there is a unique lex shortest path.  $\square$

We decompose a weighted partial 2-tree  $G$  that is not outerplanar into two weighted partial 2-trees  $G_1$  and  $G_2$  in such a way that  $LSC(G)$  is equal to the disjoint union of  $LSC(G_1)$  and  $LSC(G_2)$ . From Lemma 2.4, there exist two vertices  $u, v \in V(G)$  such that  $G - \{u, v\}$  is disconnected and has at least three components. Let  $P$  be the lex shortest path between  $u$  and  $v$  in  $G$ . By Lemma 2.3, there exist a component  $H$  in  $G - \{u, v\}$  such that  $V(P) \cap V(H) = E(P) \cap E(H) = \emptyset$ . The operation  $decomp(G, u, v)$  decomposes  $G$  into  $G_1$  and  $G_2$ , where  $V(G_1) = V(H) \cup V(P)$ ,  $E(G_1) = E(H) \cup E(P) \cup \{(x, y) \mid x \in V(H), y \in \{u, v\} \text{ and } (x, y) \in E(G)\}$ ,  $G_2 = G[V(G) \setminus V(H)]$ . An example is shown in Figure 2 to illustrate the operation  $decomp(G, u, v)$ .

We use the following notation for the rest of the paper.  $G$  is a weighted partial 2-tree that is not outerplanar.  $\{u, v\}$  is a vertex separator that disconnects  $G$  into at least three components.  $H$  is a component in  $G - \{u, v\}$  such that  $V(lsp(u, v)) \cap V(H) = \emptyset$  and  $E(lsp(u, v)) \cap E(H) = \emptyset$ .  $G_1$  and  $G_2$  are the graphs obtained from the operation  $decomp(G, u, v)$ .

From the definition of  $decomp(G, u, v)$ , we have the following two observations.

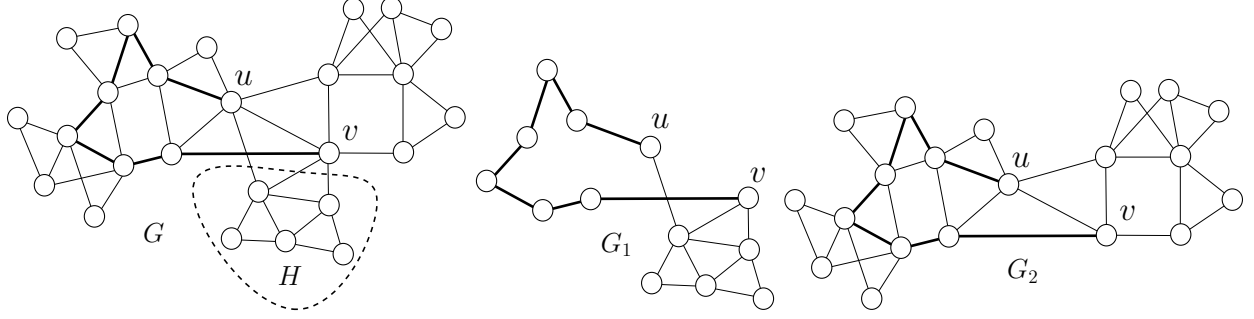


Figure 2: For a weighted partial 2-tree  $G$ ,  $lsp(u, v)$  is shown in thick edges. Also  $H \subset G$  is shown.  $G_1$  and  $G_2$  are the decomposed graphs of  $G$ .

**Observation 1** For  $x, y \in V(lsp(u, v))$ ,  $lsp(x, y)$  is in  $G_i$ .

**Proof** This observation follows, since every subpath of a lex shortest path is a lex shortest path.  $\square$

**Observation 2** For  $x \in V(G_i)$  and  $y \in V(G_i) \setminus V(lsp(u, v))$ , every path  $P(x, y)$  in  $G$  such that the internal vertices of  $P$  are in  $V(G_i) \setminus \{u, v\}$ , is present in  $G_i$ .

**Proof** Assume that  $P(x, y)$  is not in  $G_i$ . In the path  $P$  from  $x$  to  $y$ , let  $(a, b)$  be the first edge such that  $(a, b) \notin E(P)$ . Clearly,  $b \notin V(G_i)$ . It follows that  $a$  is an intermediate vertex in  $P$  and  $a \in \{u, v\}$ ; a contradiction to the premise that no intermediate vertex in  $P$  belong to  $\{u, v\}$ . Hence, the observation.  $\square$

**Lemma 2.7** For  $i \in \{1, 2\}$ , for every two vertices  $x, y \in V(G_i)$ ,  $lsp(x, y)$  is in  $G_i$ .

**Proof** If no vertex is common in  $lsp(x, y)$  and  $lsp(u, v)$ , then from Observation 2,  $lsp(x, y)$  is in  $G_i$ . If at least one vertex is common in  $lsp(x, y)$  and  $lsp(u, v)$ , then due to Lemma 2.6,  $lsp(x, y) \cap lsp(u, v)$  is  $lsp(a, b)$  for some  $a, b \in V(lsp(u, v)) \cap V(lsp(x, y))$ . The  $lsp(x, y)$  can be viewed as a union of three paths  $P(x, a)$ ,  $P(a, b)$  and  $P(b, y)$ . From Observation 1,  $P(a, b)$  is contained in  $G_i$ . If  $x = a$ , then trivially  $P(x, a)$  appears in  $G_i$ . Also, if  $y = b$ , then clearly  $P(b, y)$  appears in  $G_i$ . If  $x \neq a$ , then  $x \notin V(lsp(u, v))$ . Similarly, if  $y \neq b$ , then  $y \notin V(lsp(u, v))$ . From Observation 2, it follows that both  $P(x, a)$  and  $P(b, y)$  appear in  $G_i$ . These observations imply that  $lsp(x, y)$  is in  $G_i$ .  $\square$

**Corollary 2.8** For  $i \in \{1, 2\}$ , if there is a cycle  $C$  in  $LSC(G_i)$ , then  $C$  is in  $LSC(G)$ .

**Proof** From Lemmas 2.7 and 2.5.(a), for every  $x, y \in V(G_i)$ ,  $lsp_{G_i}(x, y)$  and  $lsp(x, y)$  are same. Since  $C \in LSC(G_i)$ , for every  $x, y \in V(C)$ ,  $lsp_{G_i}(x, y)$  is contained in  $C$ . Consequently, for every  $x, y \in V(C)$ ,  $lsp(x, y)$  is contained in  $C$ . Hence  $C \in LSC(G)$ .  $\square$

**Theorem 2.9**  $LSC(G) = LSC(G_1) \uplus LSC(G_2)$ .

**Proof** Since  $E(G_1) \cap E(G_2)$  is  $E(lsp(u, v))$ ,  $LSC(G_1)$  and  $LSC(G_2)$  are disjoint. We now prove that  $LSC(G) \subseteq LSC(G_1) \uplus LSC(G_2)$ . Let  $C \in LSC(G)$ . If  $C$  contains at most one vertex from  $\{u, v\}$ , then  $C$  is contained either in  $G_1$  or  $G_2$ , because  $\{u, v\}$  is a vertex separator. Consider the other case when  $C$  contains both  $u$  and  $v$ . Since  $C \in LSC(G)$ ,  $C$  contains  $lsp(u, v)$ . Note that  $lsp(u, v)$  is contained both in  $G_1$  and  $G_2$ . Observe that  $E(C) \setminus E(lsp(u, v))$  belongs to  $E(G_i)$  for some  $i \in \{1, 2\}$ , because  $E(G_1) \cap E(G_2) = E(lsp(u, v))$ . Hence,  $C$  is either in  $G_1$  or  $G_2$ . In both of the cases, by Lemma 2.5.(b),  $C$  is either in  $LSC(G_1)$  or  $LSC(G_2)$ . Therefore,  $LSC(G) \subseteq LSC(G_1) \uplus LSC(G_2)$ . From Corollary 2.8,  $LSC(G_1) \uplus LSC(G_2) \subseteq LSC(G)$ . Hence,  $LSC(G) = LSC(G_1) \uplus LSC(G_2)$ .  $\square$

**Lemma 2.10** The number of  $K_{2,3}$ -subdivisions in each of  $G_1$  and  $G_2$  is less than the number of  $K_{2,3}$ -subdivisions in  $G$ .

**Proof** Recall that there is a  $K_{2,3}(\{u, v\}, \{x, y, z\})$ -subdivision in  $G$ , and  $G_1$  and  $G_2$  are obtained from  $\text{decomp}(G, u, v)$ . Without loss of generality, assume that  $x \in V(H)$ . Then at most one vertex from  $\{y, z\}$  is in  $G_1$ . Further, observe that  $x$  is not in  $G_2$ . Therefore, no  $K_{2,3}(\{u, v\}, \{x, y, z\})$ -subdivision exist in  $G_1$  and  $G_2$ . Thus the lemma holds.  $\square$

### Proof of Theorem 1.1

**Proof** We use induction on the number of  $K_{2,3}$ -subdivisions in  $G$ . If the number of  $K_{2,3}$ -subdivisions in  $G$  is zero, then  $G$  is outerplanar, since  $G$  is a partial 2-tree. From Lemma 2.2,  $|LSC(G)| = m - n + 1$  when  $G$  is outerplanar. Hence base case is true. If  $G$  is not an outerplanar graph, then there exists a  $K_{2,3}(\{u, v\}, \{x, y, z\})$ -subdivision in  $G$ . From Lemma 2.4,  $G - \{u, v\}$  is disconnected and contains at least three components. Let  $P$  be the  $\text{lsp}(u, v)$  in  $G$  and  $k = |V(P)|$ . We apply  $\text{decomp}(G, u, v)$  to obtain  $G_1$  and  $G_2$  from  $G$ . For  $i \in \{1, 2\}$ ,  $m_i$  and  $n_i$  indicate  $|E(G_i)|$  and  $|V(G_i)|$ , respectively. Now, we can apply induction hypothesis due to Lemma 2.10. By induction hypothesis, it follows that  $|LSC(G_i)| = m_i - n_i + 1$  for  $i \in \{1, 2\}$ . As  $P$  is present in  $G_1$  and  $G_2$ , it follows that  $n_1 + n_2 = n + k$  and  $m_1 + m_2 = m + k - 1$ . From Theorem 2.9,  $LSC(G) = LSC(G_1) \uplus LSC(G_2)$ . Hence  $|LSC(G)| = |LSC(G_1)| + |LSC(G_2)| = m_1 - n_1 + 1 + m_2 - n_2 + 1 = m - n + 1$ . Therefore,  $|LSC(G)| = m - n + 1$ .  $\square$

### Proof of Theorem 1.2

**Proof** For a simple weighted graph  $G$ , from Lemma 2.1, an  $MCB(G) \subseteq LSC(G)$ . The cardinality of any cycle basis in a graph is known to be  $m - n + 1$ . For a weighted partial 2-tree  $G$ , by Theorem 1.1, we have  $|LSC(G)| = m - n + 1$ . Therefore, the set of lex short cycles is a minimum cycle basis in weighted partial 2-trees.  $\square$

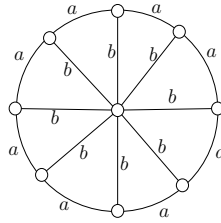


Figure 3: For the wheel graph shown, if  $b \gg a$ , then the set of all triangles and the exterior face are lex short cycles.

We present a family of partial 3-trees for which the set of lex short cycles is not a cycle basis, whose construction is as follows: Let  $G_n = K_1 + C_{n-1}$  be a wheel graph on  $n$  vertices, where  $n \geq 4$ . A wheel graph on 9 vertices is depicted in Figure 3. Note that  $G_n$  is planar. For every edge  $e \in E(G_n)$ , assign  $w(e) = a$  if  $e$  is in external face, otherwise  $w(e) = b$ , where  $a, b \in \mathbb{N}$  and  $b \gg a$ . Since every face in  $G_n$  is a lex short cycle,  $|LSC(G_n)| = m - n + 2$  by Euler's formula. Therefore,  $LSC(G_n)$  can not be a cycle basis.

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